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Stochastic Phase Lockings in a Relaxation Oscillator Forced by a Periodic Input with Additive Noise: A First-Passage-Time Approach

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Noise effects on phase lockings in a system consisting of a piecewise-linear van der Pol relaxation oscillator driven by a periodic input are studied. The problem of finding the period of the oscillator is reduced to the first-passage-time problem of the Ornstein–Uhlenbeck process with time-varying boundary. The probability density functions of the first-passage time are used to define the operator which governs a transition of an input phase density after one cycle of the oscillator. Phase lockings in a stochastic sense are investigated on the basis of the density evolution by the operator.

KEY WORDS: Additive noise; relaxation oscillation; van der Pol oscillator; first-passage time; stochastic phase locking; Ornstein–Uhlenbeck process.

1. INTRODUCTION

The influence of noise on nonlinear dynamical systems has been an object of intense investigation,[1] and the phenomenon of transitions induced by external noise has led to a revival of interest in the role of fluctuations in physical systems.[2] For instance, in a multistable system which possesses several competing states of local stability, noise can be responsible for transitions between these states. Recently "noisy" systems have received considerable attention also within the context of stochastic resonance.[3]

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The van der Pol equation

$$\frac{d^2x}{d\tau^2}+\mu(x^2-1)\frac{dx}{d\tau}+x=0 \qquad (\mu>0) \tag{1}$$

provides a typical example of a nonlinear oscillator. The original application of the system described by van der Pol arose as a model in electric circuits,[4] but a wealth of examples similar to such a system can be found in a variety of fields.[5] If the parameter μ in (1) is sufficiently large, the waveform exhibited by the van der Pol oscillator is nearly discontinuous and drastically different from those of sinusoidal oscillators. Such a discontinuous oscillation is called a relaxation oscillation.[6]

Numerous studies on forced nonlinear oscillations, including the van der Pol equation,[8] have been performed.[7] If the amplitude of the external force is sufficiently large, the forced oscillator is entrained, or phase-locked, to the external force. As the period and amplitude of the external force are changed, various patterns of $m{:}n$ phase lockings (in which the forced oscillator runs n cycles for each m cycles of the external force) and chaotic behavior appear.

Grasman and Roerdink[9] analyzed the van der Pol relaxation oscillator with additive noise. As they pointed out, if the parameter μ in (1) tends to infinity, the problem of examining the period of the system reduces to the analysis of the time necessary for a one-dimensional stochastic process to reach a boundary for the first time; hence, it is appropriate to talk of a first-passage-time approach. Although an explicit solution to such problem has not yet been obtained, an asymptotic evaluation for small noise intensity has been given.[9] To check this result, the above-mentioned authors also solved the stochastic differential equations numerically and computed the distribution of the interjump time, which is half of the period.

In this paper, we extend the first-passage-time approach[9] to analyze the system modeled by a piecewise-linear van der Pol oscillator forced by a sinusoidal input with additive noise. A transformation of the system variables leads to an Ornstein–Uhlenbeck (OU) process with a time-varying boundary. Using densities of the input phase, we define an operator which governs the transition of the density after one cycle of the oscillator. We connect the phase locking in a stochastic sense with asymptotic behavior of the density evolution.

2. PIECEWISE-LINEAR RELAXATION OSCILLATOR

Using Liénard's transformation

$$y = \int \mu(x^2 - 1)\,dx + \frac{dx}{d\tau} = \mu\left(\frac{x^3}{3} - x\right) + \frac{dx}{d\tau} \tag{2}$$

and after the change of time scale

$$\tau = t\mu, \qquad \mu = 1/\sqrt{\varepsilon} \tag{3}$$

we can rewrite Eq. (1) as the system of first-order differential equations

$$\varepsilon\dot{x} = x - x^3/3 + y \tag{4a}$$

$$\dot{y} = -x \tag{4b}$$

where a dot denotes time derivative with respect to t. Changing the right-hand side of Eq. (4a) into a piecewise-linear function, we obtain the piecewise-linear van der Pol oscillator described by

$$\varepsilon\dot{x} = \begin{cases} y - x - 5/3 & (x < -1, \quad \text{region } 1) \\ y + 2x/3 & (|x| \leqslant 1, \quad \text{region } 2) \\ y - x + 5/3 & (x > 1, \quad \text{region } 3) \end{cases} \tag{5a}$$

$$\equiv F(x, y)$$

$$\dot{y} = -x \tag{5b}$$

Figure 1 shows the limit cycle of Eq. (5) in the x–y phase plane with an N-shaped x-nullcline ($\dot{x} = 0$) and a linear y-nullcline ($\dot{y} = 0$). The local minimum $B \equiv (1, -2/3)$ and maximum $D \equiv (-1, 2/3)$ of the x-nullcline $F(x, y) = 0$ of Eq. (5) are the same as those of the x-nullcline of Eq. (4).

In the following, we shall consider the limit as $\varepsilon \to 0$. Therefore, finding the solutions of Eq. (5) becomes a singular perturbation problem. However, we shall consider only the discontinuous (or singular) solutions of Eq. (5). Setting $\varepsilon = 0$, we obtain a discontinuous approximation of Eq. (5):

$$0 = F(x, y) \tag{6a}$$

$$\dot{y} = -x \tag{6b}$$

Further, we define the set K as follows:

$$K = \{(x, y) \mid F(x, y) = 0, \partial F(x, y)/\partial x < 0\} \tag{7}$$

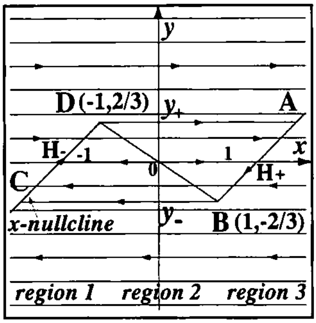

Fig. 1. The limit cycle (the closed trajectory $ABCD$) of the piecewise-linear relaxation oscillator (5) in the x–y phase plane, illustrated with an N-shaped x-nullcline ($\dot{x}=0$) and a linear y-nullcline ($\dot{y}=0$). The parameter is set to $\varepsilon = 0.001$. The vector field, shown by arrows, is almost horizontal. If a small perturbation in the horizontal direction of the phase plane displaces a state point from the limit cycle, the state point will immediately return to the limit cycle horizontally because of the large component of the vector field in the x direction. The regions $x < -1$, $|x| \leqslant 1$, and $x > 1$ are called regions 1, 2, and 3, respectively. The branches of the x-nullcline are denoted by H_- in region 1 and by H_+ in region 3.

For simplicity, the two semiinfinite lines included in K will be denoted as branches H_+ and H_-; hence, $x = y + 5/3 \equiv H_+(y)$ (region 3) and $x = y - 5/3 \equiv H_-(y)$ (region 1), which are the local solutions of $F(x, y) = 0$. From Eq. (6), we can obtain a singular solution which approximates the exact solution of Eq. (5). Let the state point $(x(t), y(t))$ of the system (5) start at time zero in an initial point $(x(0), y(0))$ in the x–y phase plane. If the initial point is not in the set K, the state point makes an instantaneous jump to a point $(x_r, y(0))$ on that branch $[x = H_+(y)$ or $x = H_-(y)]$. After the jump the state point proceeds clockwise along one branch until it reaches the point B or D; then, it leaves the branch by an instantaneous jump to the point $C \equiv (-7/3, -2/3)$ or $A \equiv (7/3, 2/3)$ located on the other branch. This means that the singular solution of the system can be described alternately by instantaneous jumps from one branch to the other and by interjump motions along them.

Singular solutions approximate the exact solutions of Eq. (5) only on K. Figure 1 illustrates the closed trajectory $ABCD$ of the singular solution and the direction of the vector field. If a small instantaneous perturbation in the x direction of the phase plane displaces a state point from the limit cycle, the state point will immediately return to it because the vector field in the x direction is very large. Hence, small instantaneous perturbations in the x direction do not affect the velocity of the state point along the x-nullcline $F(x, y) = 0$. If a continuous perturbation such as a sinusoidal function is applied to the right-hand side of Eq. (5a), then the perturbation moves the x-nullcline and thus affects the velocity along the x-nullcline. In the limit case of $\varepsilon = 0$, however, the x-direction perturbation to Eq. (5a) can be converted to the y-direction perturbation to Eq. (5b), so that only the y-direction perturbations to Eq. (5b) will be considered in the present work.

In order to calculate the period T_ε of the free oscillation, let the periodic solution start at time $t = 0$ at a point A. The time interval spent by the state point on the segment AB is given by

$$-\int_{7/3}^{1} \frac{dx}{x} = \ln 7 - \ln 3 \tag{8}$$

Because the x-nullcline $[F(x, y) = 0]$ is a symmetric function, the state point spends the same amount of time on the segment CD. Therefore, the period T_0 of the solution is

$$T_0 = -2 \int_{7/3}^{1} \frac{dx}{x} \tag{9}$$

The period T_0 is an $O(1)$ approximation of the period T_ε of the exact solution. The more precise estimate

$$T_\varepsilon = T_0 + O(\varepsilon^{2/3}) \tag{10}$$

is given in the literature.[11] An explicit formula for the singular solution $x(t)$ with the initial condition $x(0) = 7/3$ can also be easily obtained $(n = 1, 2,...)$:

$$x(t) = \begin{cases} 7/3 \exp\{-[t - (n-1)\, T_0]\}, & (n-1)\, T_0 \leqslant t < (n - 1/2)\, T_0 \\ -7/3 \exp\{-[t - (n - 1/2)\, T_0]\}, & (n - 1/2)\, T_0 \leqslant t < nT_0 \end{cases}$$

3. RELAXATION OSCILLATOR DRIVEN BY PERIODIC INPUTS WITH ADDITIVE NOISE

We analyze the system of stochastic differential equations

$$\varepsilon\, dX(t) = F(X(t), Y(t))\, dt \tag{11a}$$

$$dY(t) = \{-X(t) + V(t)\}\, dt + \delta\, dW(t) \tag{11b}$$

where, as customary, $W(t)$ denotes the standard Wiener process and δ is a positive parameter. The term $\delta\, dW(t)$ represents an additive noise, which means that it is independent of the state variables of the system. The function $V(t)$ in (11b) denotes the sinusoidal periodic input:

$$V(t) = M \sin[2\pi(t/I + \theta_0)] \tag{12}$$

where M, I, and θ_0 denote the amplitude, period, and initial phase of the input, respectively. A precise definition of "phase" will be given in Section 4.

In the limit as the parameter ε tends to zero, Eqs. (11) yield the reduced system

$$dY_-(t) = \{-H_-(Y_-(t)) + V(t)\}\, dt + \delta\, dW(t) \qquad \text{(region 1)} \quad (13a)$$

$$dY_+(t) = \{-H_+(Y_+(t)) + V(t)\}\, dt + \delta\, dW(t) \qquad \text{(region 3)} \quad (13b)$$

where

$$H_\pm(y) = y \pm 5/3 \tag{14}$$

For ε sufficiently small, the motion of a state point can be viewed as governed by Eq. (13a) in region 1 and by Eq. (13b) in region 3. Hence, the period of the oscillation equals the time spent by the state point on these regions.

We shall first analyze the stochastic trajectories on the branch H_+ in region 3. Let the state point start at time $t = 0$ at initial point A (see Fig. 1), viz. $Y_+(0) = y_+$ ($\equiv 2/3$). We consider the time interval T_+ when the one-dimensional stochastic process $Y_+(t)$ reaches a boundary set at $y = y_-$ ($\equiv -2/3$) for the first time after leaving the initial point y_+. This time interval is the random variable representing the first-passage time from state y_+ to state y_-:

$$T_+ = \inf\{t > 0\colon Y_+(t) = y_- \mid Y_+(0) = y_+\} \tag{15}$$

Similarly, the time interval T_- is defined as follows:

$$T_- = \inf\{t > 0\colon Y_-(t) = y_+ \mid Y_-(0) = y_-\} \tag{16}$$

Therefore, the period T of the stochastically perturbed oscillator is given by the random variable

$$T = T_+ + T_- \tag{17}$$

In order to eliminate the time dependence of the coefficients of the stochastic differential equations (13a) and (13b), we use the transformation

$$y' = y - \int_0^t \exp(s - t)\, V(s)\, ds \qquad (18a)$$

$$t' = t \qquad (18b)$$

on the processes $Y_+(t)$ and $Y_-(t)$. The transformed processes $Y'_+(t)$ and $Y'_-(t)$ are governed by the following stochastic differential equations (see appendix):

$$dY'_-(t) = \{-H_-(Y'_-(t))\}\,dt + \delta\,dW(t) \qquad \text{(region 1)} \tag{19a}$$

$$dY'_+(t) = \{-H_+(Y'_+(t))\}\,dt + \delta\,dW(t) \qquad \text{(region 3)} \tag{19b}$$

The constant boundaries $y = y_-$ and $y = y_+$ of $Y_+(t)$ and $Y_-(t)$ are then changed into the time-varying boundaries $y' = L_+(t)$ and $y' = L_-(t)$ for processes $Y'_+(t)$ and $Y'_-(t)$, where

$$L_\pm(t) = y_\mp - \int_0^t \exp(s-t)\,V(s)\,ds \tag{20}$$

Denoting by $f_\pm(t; y_\pm)$ the first-passage-time probability density functions (pdf's) of processes $Y_\pm(t)$ with the initial state $Y_\pm(0) = y_\pm$ and by $\hat{f}_\pm(t'; y_\pm)$ the corresponding first-passage-time pdf's of the processes $Y'_\pm(t')$, one has

$$f_\pm(t; y_\pm) = \hat{f}_\pm(t'; y_\pm)\left|\frac{dt'}{dt}\right| = \hat{f}_\pm(t'; y_\pm) \tag{21}$$

Equations (14) and (19) imply that $Y'_\pm(t)$ are the Ornstein–Uhlenbeck processes.[15] Hence, the numerical procedure proposed by Buonocore *et al.*[12] can be implemented to compute the first-passage-time pdf's in the presence of time-dependent boundaries. This is based on the numerical solution of an integral equation satisfied by the first-passage-time pdf whose kernel is a continuous function. Hence, without need to resort to simulation algorithms, one can evaluate the unknown pdf's to an arbitrary degree of accuracy.

4. STOCHASTIC PHASE LOCKING

Since the van der Pol oscillator was proposed, behaviors of the forced oscillator have been intensively investigated.[7] It is well known that the deterministic system of a periodically forced oscillator exhibits the phenomenon of phase lockings. When a driving period I is in the neighborhood of the period of the free oscillator, one may expect an entrained oscillation at the driving period; this is called a 1:1 phase locking. On the other hand, even though the period of the system is significantly different from the driving period, phase lockings occur: the period of the system is entrained to a period which is an integral multiple or submultiple of the driving period. As the period and/or amplitude of the driver are varied, the driven oscillator may show an $m{:}n$ phase locking in which the driven oscillator reiterates the cycle n times while the driver oscillates repeatedly m times.

In order to discuss phase lockings more precisely, let us define an input phase as

$$\tau(t;\theta_0)=t/I+\theta_0 \pmod 1 \tag{22}$$

where θ_0 and I denote the initial phase and period of the input $V(t)$ defined by Eq. (12), respectively. The phase τ increases at the rate of $1/I$ as time passes and is reset after reaching unity. Thus, it is regarded as the time normalized by the input period I, taking values on the unit circle $S=[0, 1]$.

Figure 2a illustrates the waveforms of a sinusoidal forcing term and the variable x of the system (11) deprived of the noise ($\delta=0$) when a 1:1 phase locking occurs. In this case, whenever a state point of the system returns to the point $A\equiv(7/3, 2/3)$ indicated in Fig. 1, which hereafter will be called the "base point," one can observe the same input phase. Figure 2b shows an example of a 1:3 phase locking. At the base point, three distinct input phases are in turn observed repeatedly.

Figure 3 illustrates the waveforms of the variable x of the system (11) and forcing term with additive noise. In Fig. 3a, the noise intensity is small ($\delta=0.01$) as compared to the amplitude of the forcing term ($M=1.0$) and,

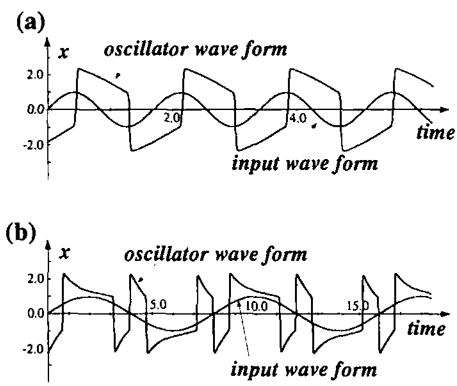

Fig. 2. Waveforms of $x(t)$ and of input $V(t)$ for the system (11) in the absence of noise ($\varepsilon=0.001$, $\delta=0.0$). (a) A 1:1 phase locking, in which the forced oscillator runs one cycle for each cycle of the sinusoidal input. Every time a state point returns to the base point ($x=7/3$), the same input phase is observed. Input parameters are $I=1.7$ and $M=1.0$. (b) A 1:3 phase locking, in which the forced oscillator runs three cycles for each cycle of the input. At the base point, three distinct input phases are in turn observed repeatedly. The input parameters are $I=8.0$ and $M=1.0$.

in Fig. 3b, the intensity of noise is one-tenth of the amplitude of the forcing term ($\delta = 0.1$). Every time a state point gets back to the base point, a variety of values of the input phase can be observed. In Fig. 3c an example of a 1:3 phase locking is illustrated when the intensity of the noise δ is 0.1. For one oscillation of the driving term, the oscillator oscillates three times. In contrast to the deterministic case, each value of the three input phases at the base point slightly varies in every third rotation.

In the presence of noise (see Fig. 3), the period of the self-sustained oscillation seems to fall into synchronism with the driving period even though the period of the oscillation is varied stochastically and slightly differs from the entrained period of the deterministic system. It is appropriate to refer to this phenomenon as stochastic phase locking, in analogy with its deterministic counterpart. In the following, a more detailed discussion of stochastic phase locking will be provided.

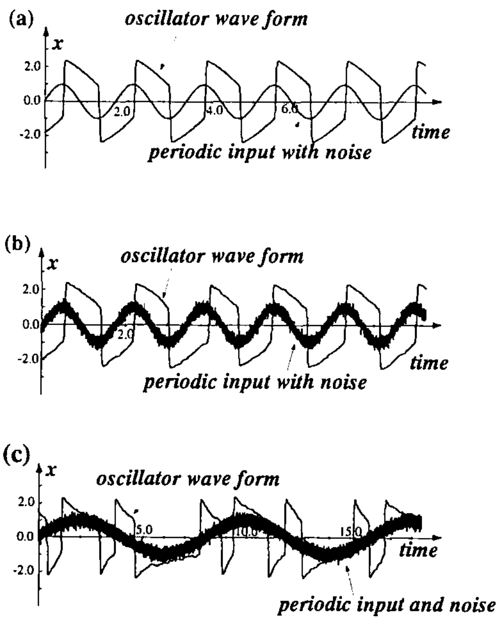

Fig. 3. Examples of stochastic phase lockings. In all cases, $M = 1.0$ and $\varepsilon = 0.001$. Both (a) and (b) show 1:1 stochastic phase lockings. (a) The noise intensity is small as compared to the amplitude of the driving term ($I = 1.7$, $\delta = 0.01$). (b) The noise intensity is one-tenth of the amplitude of the driving term ($I = 1.7$, $\delta = 0.1$). Every time a state point gets back to the base point A (whose abscissa is $x = 7/3$), in contrast to the deterministic case shown in Fig. 2a, a variety of values of the input phase can be observed. (c) An example of a stochastic 1:3 phase locking ($I = 8.0$, $\delta = 0.1$). For each oscillation of the input, the oscillator appears to oscillate three times. Each value of the three input phases at the base point varies in every third rotation.

5. OPERATORS AND PDF'S OF THE INPUT PHASE

Let the state point governed by the system (11) be at the base point A (see Fig. 1) at time $t=0$ with a particular initial phase θ_0 of the forcing term $V(t)$. If it reaches the point B at time $t=t_+$, the increment of the phase is t_+/I (mod 1). Therefore, at that time the input phase is expressed as $\theta'_0=\tau(t_+;\theta_0)=t_+/I+\theta_0$ (mod 1). It is clear that the input phase θ'_0 is a random variable because it depends on the random variable T_+. When the state point reaches the point B in region 3, as mentioned before, it makes an instantaneous jump to the point C in region 1. Hence, at point C the input phase of the forcing term is also θ'_0. Let the state point spend a time interval t_- while it proceeds from C to D. According to the above procedure, the input phase at point D can be expressed as $\theta_1=\tau(t_++t_-;\theta_0)=(t_++t_-)/I+\theta_0$ (mod 1). This means that after the state point returns to the base point A, the input phase changes from θ_0 into θ_1.

Let now $f_T(t;\theta_0)$ be the probability density function (pdf) of the time interval T corresponding to one cycle of the trajectory starting at the base point with an initial input phase θ_0. The time interval T corresponds to the period of the system (11) and is always measured on the basis of the base point A. Let $f_+(t;\theta_0)$ and $f_-(t;\theta'_0)$ denote the first-passage-time pdf's in regions 3 and 1 with the corresponding initial input phases θ_0 and θ'_0 of the forcing term, respectively. The pdf of the time interval T is given by the convolution of $f_-(t;\theta'_0)$ with $f_+(t;\theta_0)$ as the initial input phase θ'_0 in region 1 is changed depending on the time interval t_+ spent by the state point in region 3:

$$f_T(t;\theta_0)=\int_0^\infty f_-(t-t_+;\theta'_0)\,f_+(t_+;\theta_0)\,dt_+ \tag{23}$$

$$\theta'_0=\tau(t_+;\theta_0) \quad (\text{mod } 1) \tag{24}$$

In order to describe the motion of a state point in terms of the input phase, we transform the pdf of T into that of the input phase as follows:

$$g(\theta;\theta_0)=\sum_{\tau(t_i;\theta_0)=\theta}\frac{f_T(t_i;\theta_0)}{|\tau'(t_i;\theta_0)|} \tag{25}$$

where $\tau'(t;\theta)$ means $\partial\tau(t;\theta)/\partial t$. So far we have considered the fixed initial input phase at the base point. We shall now deal with the case when the initial phase θ_0 is distributed on $S=[0,1]$ according to a probability density function $h_0(\theta_0)$. Let us denote by $h_1(\theta)$ the pdf of the input phase after a state point returns to the base point in one cycle with the initial phase pdf $h_0(\theta_0)$. Hence,

$$h_1(\theta) = \int_S g(\theta; \theta_0)\, h_0(\theta_0)\, d\theta_0 \tag{26}$$

or $h_1 = \mathrm{P}h_0$, where P is the operator defined by

$$\mathrm{P}h_0(\theta) = \int_S g(\theta; \theta_0)\, h_0(\theta_0)\, d\theta_0 \tag{27}$$

For any given pdf h_0, making use of operator P, it is possible to define h_n inductively:

$$h_n = \mathrm{P}h_{n-1} = \mathrm{P}(\mathrm{P}h_{n-2}) = \cdots = \mathrm{P}^n h_0 \tag{28}$$

We conclude this section with some definitions necessary for the following discussion. By $L^1(S)$ we shall denote the class of functions f on the circle S such that

$$\|f\| = \int_S |f(x)|\, dx < \infty \tag{29}$$

Then, $\|f\|$ is the $L^1(S)$ norm of f. As is customary, we say that $h \in L^1(S)$ is a density if h is nonnegative and its integral over the domain S is equal to unity. Let the set $\mathscr{D}$ of pdf's be defined by

$$\mathscr{D} = \{h \in L^1(S);\, h \geqslant 0,\, \|h\| = 1\} \tag{30}$$

If P is the operator defined above and if

$$\mathrm{P}h^* = h^* \tag{31}$$

for some $h^* \in \mathscr{D}$, then h^* is called an invariant density of the operator P.

We can now define[13] the asymptotic stability of a sequence of density functions $\{\mathrm{P}^n h_0\}$ in the following way: For every $h_0 \in \mathscr{D}$, a sequence $\{\mathrm{P}^n h_0\}$ is said to be asymptotically stable if there exists a unique invariant density h^* and

$$\lim_{n \to \infty} \|\mathrm{P}^n h_0 - h^*\| = 0 \tag{32}$$

6. NUMERICAL ANALYSIS

In this section we shall provide two examples of pdf evolution which display asymptotic stability and discuss the stochastic phase locking in terms of pdf evolution. In order to obtain the pdf's of the input phase, we have made use of a specific numerical procedure[12] to calculate first-passage-time pdf's through time-varying boundaries for diffusion processes. For a given initial density h_0 of the input phase, the pdf sequence $\{P^n h_0\}$ of the input phase can be numerically calculated in accordance with the method outlined in Section 5. The first example presents an asymptotic stable sequence of pdf's showing a stochastic 1:1 phase locking in terms of density evolution. The second example, exhibiting a stochastic 1:3 phase locking, indicates that the behaviors of the pdf sequences that converge to their invariant densities are quite different from the case of the first example.

6.1. 1:1 Phase Locking

Figure 4 illustrates the numerically obtained results on the asymptotic stability of the sequence of input phase pdf's. This corresponds to the stochastic 1:1 phase locking shown in Fig. 3. Figures 4a, 4c, and 4e show the initial densities and Figs. 4b, 4d, and 4f the corresponding evolution of pdf's. For simplicity, only h_1, h_2, h_{100}, h_{101}, and h_{102} have been plotted, while other transient pdf's have been omitted. From the figure we see that the relations $Ph_{100} = h_{101}$ and $Ph_{101} = h_{102}$ hold to a very high degree of accuracy. Moreover, the figure shows that in each evolution of pdf's from distinct initial density functions the sequences of pdf's seem to converge to the same density h^* with one sharp peak at a certain phase, indicating that a stochastic 1:1 phase locking occurs. The sequence $P^n h_0$ quickly converges to the density h^* independent of the initial density h_0. Thus, we are led to conjecture that the operator P has a unique invariant density h^* and that the sequence of pdf's $\{P^n h_0\}$ is asymptotically stable.

6.2. 1:3 Phase Locking

Figures 5 and 6 illustrate other examples of asymptotic stability of the sequence $\{P^n h_0\}$ of pdf's. This case corresponds to the stochastic 1:3 phase locking depicted in Fig. 3. Figure 5 illustrates pdf's h_{100}, h_{101}, h_{102}, and h_{103} when the initial density function of the input phase is uniformly distributed on S. The operator P transforms functions g_{a1}, g_{a2}, and g_{a3} (Fig. 5a) to functions g_{b2}, g_{b3}, and g_{b1} (Fig. 5b), and the second iterate of the operator P yields g_{c3}, g_{c1}, and g_{c2} (Fig. 5c). By the third iterate of the operator, we obtain g_{d1}, g_{d2}, and g_{d3}, which are, respectively, identical to g_{a1}, g_{a2}, and g_{a3} to a very high degree of accuracy. Figure 6 shows densities $P^{100}h_0$ through $P^{103}h_0$ with a different initial density, having omitted the transient consisting of 99 densities. Although from Figs. 5 and 6, $P^{103}h_0$ and $P^{100}h_0$

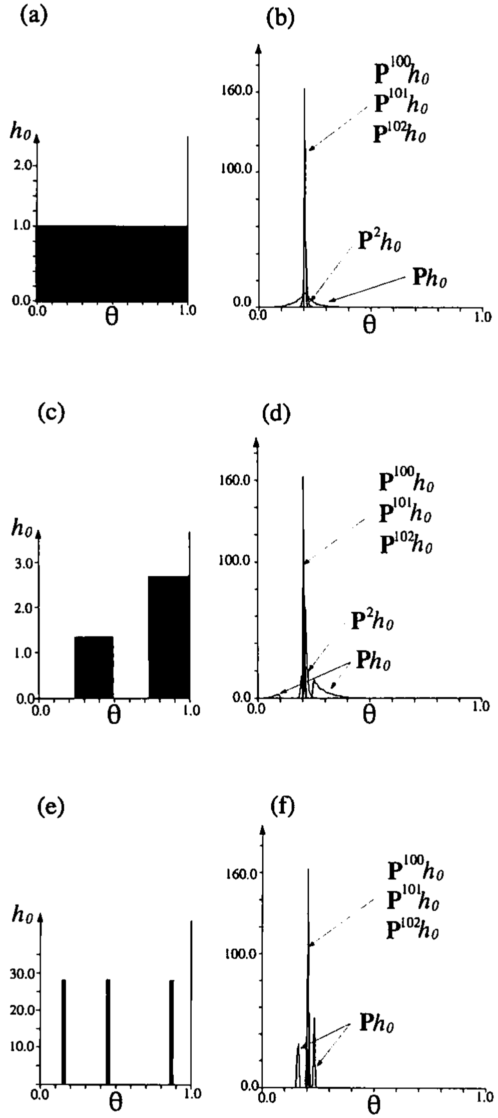

Fig. 4. Numerical illustration of the asymptotically stable sequence $\{P^n h_0\}$. This corresponds to the stochastic 1:1 phase locking shown in Fig. 3a. (a, c, e) The initial densities. (b, d, f) The densities Ph_0, P^2h_0, $P^{100}h_0$, $P^{101}h_0$, and $P^{102}h_0$ starting from initial densities in (a), (c), and (e), respectively. Other transient densities have been omitted. The sequence $\{P^n h_0\}$ seems to attain a unique density which is independent of the initial densities. The parameter values are $I = 1.7$, $M = 1.0$, and $\delta = 0.01$.

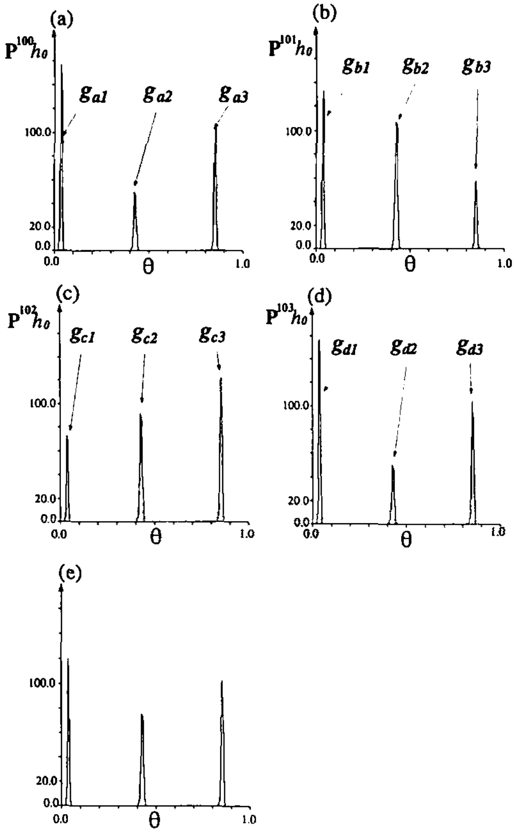

Fig. 5. Another example of the asymptotic stability of the pdf sequence $\{P^n h_0\}$. This corresponds to a 1:3 phase locking in the presence of additive noise shown in Fig. 3c. Pdf's $P^{100}h_0$, $P^{101}h_0$, $P^{102}h_0$, and $P^{103}h_0$ are illustrated starting from a uniform density. The operator P transforms functions g_{a1}, g_{a2}, and g_{a3} in (a) into the functions g_{b2}, g_{b3}, and g_{b1} in (b), and the second iterate of the operator P yields g_{c3}, g_{c1}, and g_{c2} shown in (c). By the third iterate of the operator, we obtain g_{d1}, g_{d2}, and g_{d3}, which are almost, but not exactly, identical to g_{a1}, g_{a2}, and g_{a3}, respectively. (e) An invariant density, exhibited after a very long transient ($P^n h_0$, $n = 10^6$). Densities have been numerically computed by dividing the interval [0, 1] on the abscissa into 800 equal bins. The parameter values are $I = 8.0$, $M = 1.0$, and $\delta = 0.01$.

look alike, they are not exactly such; indeed, in both cases $\{P^n h_0\}$ converges to an invariant density after a very long transient (Fig. 5e). Let now g_i $(i = 1, 2, 3) \in \mathscr{D}$ be normalized functions obtained from g_{a1}, g_{a2}, and g_{a3}, respectively:

$$g_i = g_{ai}/\|g_{ai}\| \tag{33}$$

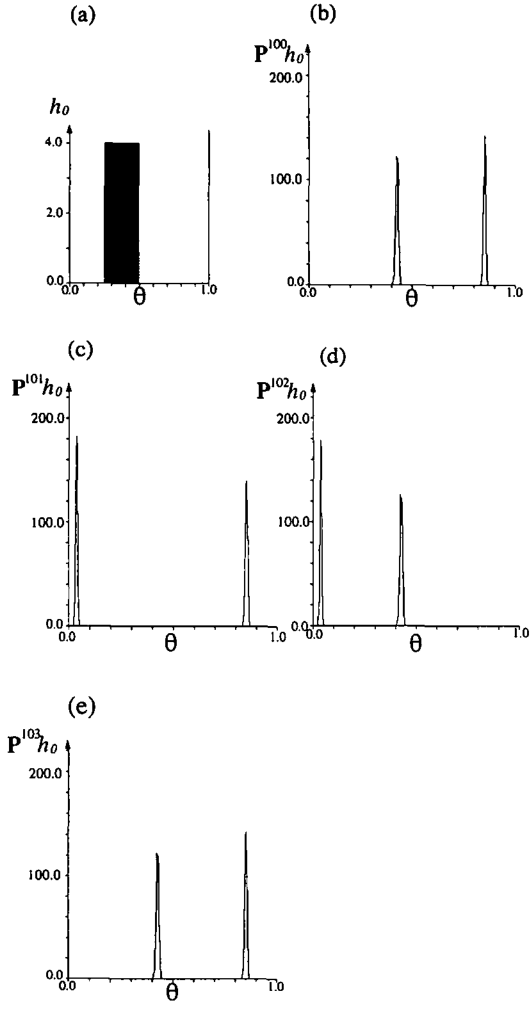

Fig. 6. The strong dependence of the transient densities on the initial density. (a) The initial density. Parameter values are the same as those of Fig. 5. Though the pdf sequence seems to exhibit a periodicity with period three, the sequence evolves to the unique invariant density shown in Fig. 5e after a long transient.

Then, from Fig. 5, we see that the following approximate relation holds:

$$P^3 g_i \approx g_i \qquad (i = 1, 2, 3) \tag{34}$$

Hence, it is appropriate to call g_i $(i = 1, 2, 3)$ pseudoeigenfunctions (of P^3). Note that in a strict sense the operator P^3 does not have any such eigenfunction other than the unique invariant density shown in Fig. 5e (cf. Section 7). The numerical examples shown in Figs. 5 and 6 lead us to describe approximately the densities $\{P^n h_0\}$ $(n > N)$, for any initial density $h_0 \in \mathscr{D}$ and large N, by a linear combination of the pseudoeigenfunctions g_i $(i = 1, 2, 3)$:

$$\begin{gathered} P^n h_0 \approx \alpha_n(h_0)\, g_1 + \beta_n(h_0)\, g_2 + \gamma_n(h_0)\, g_3 \\ \alpha_n(h_0) + \beta_n(h_0) + \gamma_n(h_0) = 1 \end{gathered} \tag{35}$$

where nonnegative coefficients $\alpha_n(h_0)$, $\beta_n(h_0)$, and $\gamma_n(h_0)$ depend on the initial density h_0 and the iteration time n. Since $Pg_1 \approx g_2$, $Pg_2 \approx g_3$, and $Pg_3 \approx g_1$,

$$P(\tfrac{1}{3} g_1 + \tfrac{1}{3} g_2 + \tfrac{1}{3} g_3) \approx \tfrac{1}{3} g_1 + \tfrac{1}{3} g_2 + \tfrac{1}{3} g_3 \tag{36}$$

which means that $g^* = (g_1 + g_2 + g_3)/3$ is the approximate invariant density. As is seen from Fig. 5e and the conjecture in Section 7, the sequence $\{P^n h_0\}$ will evolve to its invariant density. Hence, the coefficients $\alpha_n(h_0)$, $\beta_n(h_0)$, and $\gamma_n(h_0)$ must approach 1/3 as the iteration time n increases. However, as is seen from Fig. 5, the speed of convergence to the invariant density of the system is very small; furthermore, such convergence significantly differs from that of example (i) even though both examples exhibit a unique asymptotically stable invariant density.

7. DISCUSSION

The operator P is defined by the kernel $g(\theta; \theta_0)$ in Eq. (27). Since

$$g(\theta; \theta_0) \geqslant 0, \qquad \int_S g(\theta; \theta_0)\, d\theta = 1 \tag{37}$$

this is a stochastic kernel, apparently possessing the following property:

$$\int_S \inf_{\theta_0} g(\theta; \theta_0)\, d\theta > 0 \tag{38}$$

Thus the operator P always has a unique asymptotically stable invariant density (cf., for instance, Corollary 5.7.1 of ref. 13). This property, however, has not been proved, since the kernel is not known analytically. Indeed, we have associated the density evolution of the system to a stochastic phase locking, and we have visualized its evolution by means of numerical calculations. To this purpose the term asymptotic stability has been used even though a mathematical proof is still lacking.

We have thus proposed a method which enables us to analyze numerically the phenomenon of stochastic phase lockings through density evolution without having to simulate stochastic differential equations in order to construct histograms of the periods of the system. We have also presented two examples of stochastic phase lockings, a 1:1 phase locking and a 1:3 phase locking. In the latter example, the evolution of the density sequence $\{P^n h_0\}$ had a very long transient and is apparently much different from the former example. We gave a crude discussion on this long transient using the pseudoeigenfunctions g_i which were arbitrarily defined by a numerical computation. The long transient of the density evolution is caused by the smallness of the noise. Thus the pseudoeigenfunctions can be approximated by a Gaussian distribution and we can evaluate the convergence speed of the density evolution to an invariant density. We have postponed these detailed discussions since the main purpose of the present paper is to propose a method for the analysis of stochastic phase lockings based on an operator which governs a density evolution.

In the absence of noise, the forced relaxation oscillator (5) is able to show various bifurcation phenomena as input period and/or amplitude vary.[14] Using the method proposed in this paper, the effect of noise on such bifurcations can be analyzed.

We finally remark that, as is well known, fluctuations in physical systems can be accounted for either in terms of stochastic systems or by resorting to deterministic models that exhibit chaotic behavior. In the present paper, our attention has been focused on the former. A study of the effect of noise in terms of deterministic systems exhibiting chaotic behavior is certainly an interesting and challenging task. This, however, will be the object of future investigations.

APPENDIX

In order to make use of the numerical procedure of ref. 12 to evaluate first-passage-time pdf's, preliminarily a transformation is applied to the system (13). Let

$$p_{\pm}(z, \tau \mid y, t) \equiv \frac{\partial}{\partial z} \Pr\{Y_{\pm}(\tau) \leq z \mid Y_{\pm}(t) = y\} \tag{A1}$$

denote the transition pdf's of the processes governed by the stochastic differential equations (13). The indices "−" and "+" indicate regions 1 and 3, respectively.

From Eq. (13), the Kolmogorov (or "backward") equations follow:

$$\frac{\partial}{\partial t} p_{\pm}(z, \tau \mid y, t)$$
$$= \{H_{\pm}(z) + V(t)\} \frac{\partial}{\partial y} p_{\pm}(z, \tau \mid y, t) + \frac{\delta}{2} \frac{\partial^2}{\partial y^2} p_{\pm}(z, \tau \mid y, t) \quad \text{(A2)}$$

Applying the transformation

$$y' = \psi(y, t) = y - \int_0^t \exp(s - t)\, V(s)\, ds \quad \text{(A3a)}$$

$$t' = \phi(t) = t \quad \text{(A3b)}$$

to Eq. (A2), we find

$$\frac{\partial}{\partial t'} p_{\pm}(z, t' \mid y', 0)$$
$$= \frac{1}{\phi'(t)} \left[-\frac{\partial \psi(y, t)}{\partial t} + H_{\pm}(z) \frac{\partial \psi(y, t)}{\partial y} + \frac{\delta}{2} \frac{\partial^2 \psi(y, t)}{\partial y^2} \right] \frac{\partial}{\partial y'} p_{\pm}(z, \tau \mid y', t')$$
$$+ \frac{\delta}{2\phi'(t)} \left(\frac{\partial \psi(y, t)}{\partial y} \right)^2 \frac{\partial^2}{\partial y'^2} p_{\pm}(z, \tau \mid y', t') \quad \text{(A4)}$$

After calculating the differential coefficients of $\phi(y, t)$ and $\psi(t)$, we finally obtain

$$\frac{\partial}{\partial t'} p_{\pm}(z, \tau \mid y', t') = H_{\pm}(z) \frac{\partial}{\partial y'} p_{\pm}(z, \tau \mid y', t') + \frac{\delta}{2} \frac{\partial^2}{\partial y'^2} p_{\pm}(z, \tau \mid y', t') \quad \text{(A5)}$$

The stochastic processes governed by the backward equations (A5) are equivalent to the processes $Y'_{-}(t)$ and $Y'_{+}(t)$ of Eqs. (19).

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